Problem set 8

Problem 1. *Prove that two permutations are congruent in $\mathbb{S}_n$ if and only if for every k they have the same number of cycles of lengths k.*

Problem 2. *Prove that with the help of a ruler only it is impossible to divide a given segment in halves.*

Problem 3 (Desargue's theorem)**.** *Lines a, b, c intersect at one point O. In triangles $A_1B_1C_1$ and $A_2B_2C_2$, vertices A_1 and A_2 lie on line a; B_1 and B_2 lie on line b; C_1 and C_2 lie on line c. Let A, B, C be the intersection points of lines B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2 respectively. Prove that points A, B, C lie on one line.*

Problem 4. *Prove that the locus of the intersection points of quadrilaterals $ABCD$ whose sides AB and CD belong to two given lines l_1 and l_2 and sides BC and AD intersect at a given point P is a line passing through the intersection point Q of lines l_1 and l_2.*

Problem 5. *Let O be the intersection point of the diagonals of quadrilateral $ABCD$; let E (resp. F) be the intersection point of the continuations of sides AB and CD (resp. BC and AD). Line EO intersects sides AD and BC at points K and L, respectively, and line FO intersects sides AB and CD at points M and N, respectively. Prove that the intersection point X of lines KN and LM lies on line EF.*

Problem 6. *Prove that:*
a)Prove that for $\alpha \in (0, \frac{\pi}{2})$

$$cosec(\alpha) > \frac{1}{\alpha} > tg(\alpha).$$

Here $cosec(x) = \frac{1}{sin(x)}$.
b)Prove that equation

$$\binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} + \dots$$

has roots

$$ctg^2\left(\frac{\pi}{2m+1}\right),\ ctg^2\left(\frac{2\pi}{2m+1}\right),\ ctg^2\left(\frac{3\pi}{2m+1}\right),\ \dots,\ ctg^2\left(\frac{m\pi}{2m+1}\right).$$

c) Prove that

$$ctg^2\left(\frac{\pi}{2m+1}\right) + ctg^2\left(\frac{2\pi}{2m+1}\right) + ctg^2\left(\frac{3\pi}{2m+1}\right) + \dots + ctg^2\left(\frac{m\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

d)Prove that

$$cosec^2\left(\frac{\pi}{2m+1}\right) + cosec^2\left(\frac{2\pi}{2m+1}\right) + cosec^2\left(\frac{3\pi}{2m+1}\right) + \dots + cosec^2\left(\frac{m\pi}{2m+1}\right) = \frac{m(2m+2)}{3}.$$

e) Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots = \frac{\pi^2}{6}.$$