Problem set 7

Exercise 1. *Prove that the number of even permutations in* $\mathbb{S}_n$ *equals to the number of odd permutations.*

Exercise 2. *Suppose that a finite group* G *has a subgroup* H *such that* $|G| = 2|H|$. *Prove that* H *is normal.*

Definition 0.1. *Two elements* g_1 *and* g_2 *in the group* G *are conjugate if there exists an element* $h \in G$, *such that* $g_1 = hg_2h^{-1}$.

Exercise 3.
a) Prove that being conjugate is an equivalence relation. Classes of equivalence are called "conjugacy classes".
b) Prove that a subgroup is normal if and only if it is a union of conjugacy classes.
c) Find all conjugacy classes and normal subgroups of $\mathbb{S}_4$.

Exercise 4. *a) Prove that for any group* G *and an element* $g \in G$ *the map*

$$\varphi\colon G \longrightarrow G$$

given by the formula

$$x \longrightarrow gxg^{-1}$$

is an isomorphism. It is called an inner automorphism.
b) Prove that a composition of two inner automorphisms of G *is an inner automorphism. Prove that inner automorphisms form a group.*
c) Find the group of inner automorphisms of $\mathbb{S}_3$.

Problem 1. *Given line* l, *a circle and points* M, N *that lie on the circle and do not lie on* l. *Consider map* P *of line* l *to itself; let* P *be the composition of the projection of* l *to the given circle from point* M *and the projection of the circle to* l *from point* N. *(If point* X *lies on line* l, *then* $P(X)$ *is the intersection of line* NY *with line* l, *where* Y *is the distinct from* M *intersection point of line* MX *with the given circle.) Prove that* P *is a projective transformation.*

Problem 2 (Sylvester's theorem). *Consider* n *lines in the plane, not all of which pass through the same point. Prove that there exists a point in which exactly two of the lines intersect.*

Problem 3. *Consider a finite projective plane. Suppose that there exists a point, contained in* $q + 1$ *lines. a) Let* l *be a line and* p *be a point not on* l. *Construct a bijection between the points on* l *and lines through* p.
b) Prove that all lines have exactly $q + 1$ *points and every point is contained in* $q + 1$ *lines.*
c) Prove that the number of points equals to the number of lines and equals to $q^2 + q + 1$.
d) Draw a picture of $\mathbb{P}^2_{\mathbb{F}_3}$ *and show it to me.*