Problem set 6

Exercise 1. *Let G be a finite group with even number of elements. Prove that the number of elements of order 2 is odd.*

Problem 1. *Denote by $\mu(n)$ the sum of all primitive $n-$th roots of unity.*
a) Compute $\mu(n)$ for $n \leqslant 8$.
b) Prove that if n_1 and n_2 are coprime, then

$$\mu(n_1 n_2) = \mu(n_1)\mu(n_2).$$

b) Compute

$$\sum_{d|n} \mu(d).$$

d) Compute $\mu(n)$.

Problem 2. *A snail is crawling with constant speed. Every 15 minutes it turns left or right. Prove that it can return back to its original position only after an integer number of hours.*

Problem 3. *a) Find a factorization of the expression*

$$a^3 + b^3 + c^3 - 3abc.$$

b) Find a formula for the roots of a cubic equation

$$x^3 + px + q = 0.$$

Problem 4. *Let $Inv(\pi)$ be the number of inversions in a permutation π. Prove that*

$$\sum_{\pi \in \mathbb{S}_n} x^{Inv(\pi)} = (1+x)(1+x+x^2)\ldots(1+x+x^2+\ldots+x^{n-1}).$$

Problem 5 (Erdős-Szekeres theorem). *Prove that for any $n, m \in \mathbb{N}$, every sequence of $nm+1$ distinct real numbers contains an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$.*