Problem set 5

Exercise 1. *Construct a surjective homomorphism from $\mathbb{S}_4$ to $\mathbb{S}_3$ (a surjective map $f\colon \mathbb{S}_4 \longrightarrow \mathbb{S}_3$, such that $f(\sigma\tau) = f(\sigma)f(\tau)$).*

Exercise 2. *a)Prove that every Möbius transformation in $PGL_2(\mathbb{C})$ has one or two fixed points. The usual formulation of this fact is that there are two fixed points counted with multiplicity.*
b)Prove that a square of a transformation $x \longrightarrow \frac{ax+b}{cx+d}$ is the identity if and only if $a + d = 0$.

Exercise 3. *a)Suppose that for a Möbius transformation $f \in PGL_2(\mathbb{C})$ there exists a point a such that $f(a) \neq a$, but $f(f(a)) = a$. Prove that f is an involution.*
b)Prove that every Möbius transformation can be presented as a composition of at most three Möbius involutions.

Problem 1 (Euler's function). *Let $n = p_1^{\alpha_1} p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime factorization of n and $\varphi(n)$ be the number of integers from 1 to n, which are coprime to n. Prove that*

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right).$$

Definition 0.1. *Consider a group*

$$\mu_n = \{z \in \mathbb{C} | z^n = 1\}$$

of roots of unity. An $n-$th root of unity is called primitive, if it generates μ_n as a group.

Exercise 4. *Prove that primitive $n-$th roots of unity are numbers $e^{\frac{2\pi k}{n}}$ for $(k, n) = 1$. Conclude that the order of μ_n is given by the Euler function $\varphi(n)$.*

Problem 2.
a)Prove that Euler's function is multiplicative, i. e. for coprime numbers n and m

$$\varphi(nm) = \varphi(n)\varphi(m).$$

b)Prove the following identity:

$$\sum_{d|n} \varphi(d) = n.$$

c)Find a new proof of the fact that for $n = p_1^{\alpha_1} p_1^{\alpha_1} \dots p_k^{\alpha_k}$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right).$$

Problem 3 (Sam Lloyd's 15 puzzle)**.**
Suppose that numbers $1, 2, 3, \dots, 15$ and "blank" are arranged in a 4×4 grid. A legal move is to swap the empty cell ("blank") with an adjacent cell. Prove that it is not possible to go through legal moves from

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	

to

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

.

Problem 4. *Circles $S_1, S_2, \dots, S_n$ are tangent to two circles R_1 and R_2. Moreover, S_1 is tangent to S_2 at A_1, S_2 is tangent to S_3 at A_2, ..., S_{n-1} is tangent to S_n at A_{n-1}. Prove that points $A_1, A_2, \dots, A_n$ lie on the same circle.*