Problem set 3

Exercise 1. *Consider a group G and its subgroup H. We call two elements x and y equivalent ($x \sim y$) if $xy^{-1} \in H$.*
1.Prove that $\sim$ is an equivalence relation. The corresponding classes of equivalence are called left cosets.
2.Prove that all left cosets have the same number of elements.
3.Prove that if $|G|$ is finite, then $|H|$ divides $|G|$.

Exercise 2. *Prove that*
a) Similarities preserve angles and send circles to circles.
b) A composition of two homotheties with coefficients λ_1, λ_2 with $\lambda_1\lambda_2 \neq 1$ is a homothety with coefficient $\lambda_1\lambda_2$.
c) If a composition of three homotheties is the identity map, then there centers lie on the same line.
d) [**Monge's theorem**] *Outer tangent lines to the circles S_1 and S_2, S_2 and S_3, S_3 and S_1 intersect in the points A, B and C respectively. Prove that points A, B and C lie on the same line.*

Exercise 3. *Let z be the point of the intersection of two tangent lines at ζ and ζ' to the circle of radius one with the center at 0. Prove that*

$$z = \frac{2}{\frac{1}{\zeta} + \frac{1}{\zeta'}}.$$

This expression is called the harmonic mean.

Problem 1 (Newton's theorem)**.** *The midpoints of the diagonals in a quadrilateral circumscribed about a circle are collinear with the center of the circle.*

Problem 2. *Let S be a set of $n+1$ integers from 1 to $2n$. Prove that at least two elements in S are coprime.*

Problem 3. *Suppose that a regular $n-$gon with vertices $A_1, \ldots, A_n$ is inscribed in a unit circle. Prove that*

$$\prod_{i=2}^{n} |A_1A_i| = n.$$