Problem set 1

Exercise 1.
1. Prove that the composition of a parallel transport and a central symmetry (in any order) is a central symmetry.
2. Prove that if one reflects a point symmetrically over points O_1, O_2, O_3 and then reflects it symmetrically over the same points once again, the point returns back to its initial position.
3. Consider three lines a, b, c on the plane. Let $F = S_a \circ S_b \circ S_c$. Prove that $F \circ F$ is a parallel transport.

Exercise 2.
1. Prove that the following identities hold in any group G:

$$(g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1},$$
$$(g_1 \cdot g_2 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_2^{-1} \cdot g_1^{-1}.$$

2. Write down multiplication table for the group of isometries preserving a square.
3. Write down multiplication table for the group of isometries preserving a rhombus which is not a square.
4. Prove that if in the group G the square of every element is the identity, then G is abelian.

Exercise 3. *Suppose that G is a group and $S \subset G$ is a finite subset, such that if $x, y \in S$, then $xy \in S$. Prove that S is a subgroup of G.*

Exercise 4 (Ptolemy's theorem). *If a quadrilateral is inscribable in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.*
Hint: use complex numbers for that.

Problem 1. *Consider two lines in the plane with the angle γ between them and suppose a grasshopper is jumping from one line to the other. Every jump is exactly 30 inches long, and the grasshopper jumps backwards whenever it has no other options. Prove that the sequence of its jumps is periodic if and only if $\frac{\gamma}{\pi}$ is a rational number.*

Problem 2. *Consider a regular polygon with vertices $A_1, A_2, \ldots, A_n$ and center O. Prove that*

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = 0.$$

Problem 3 (Lagrange theorem). *Let G be a group of finite order. Prove that for every element g*

$$g^{|G|} = 1.$$

Problem 4. *We will consider a "disease" that infects cells of a chessboard. Initially some number of the 64 cells are infected. Subsequently the infection spreads according to the following rule: if at least two neighbors of a cell are infected then the cell gets infected. (Neighbors share an edge, so each cell has at most four neighbors.) No cell is ever cured. What is the minimum number of cells one can initially infect so that the whole board is eventually infected? It is easy to see that 8 is sufficient in many ways. Prove that 7 is not enough.*