Problem set 9

Exercise 1. *a)Prove that every Möbius transformation in* $PGL_2(\mathbb{C})$ *has one or two fixed points. The usual formulation of this fact is that there are two fixed points counted with multiplicity.*
b)Prove that a square of a transformation $x \longrightarrow \frac{ax+b}{cx+d}$ *is the identity if and only if* $a + d = 0$.

Exercise 2. *a)Suppose that for a Möbius transformation* $f \in PGL_2(\mathbb{C})$ *there exists a point* a *such that* $f(a) \neq a$, *but* $f(f(a)) = a$. *Prove that* f *is an involution.*
b)Prove that every Möbius transformation can be presented as a composition of at most three Möbius involutions.

Exercise 3. *Let* V *be a space of polynomials of degree not greater than* d. *Prove that for every* $a_1, a_2, \dots a_d \in \mathbb{R}$ *the set of polynomials*

$$\frac{(x-a_1)(x-a_2)\dots\widehat{(x-a_i)}\dots(x-a_n)}{(a_i-a_1)(a_i-a_2)\dots\widehat{(a_i-a_i)}\dots(a_i-a_n)}, \qquad 1 \leqslant i \leqslant n$$

forms a basis of V.

Definition. *An* $n-th$ *root of unity is called primitive, if it generates* μ_n *as a group.*

Exercise 4. *Prove that primitive* $n-th$ *roots of unity are numbers* $e^{\frac{2\pi k}{n}}$ *for* $(k, n) = 1$. *Conclude that the order of* μ_n *is given by the Euler function* $\varphi(n)$.

Problem 1.
a)Prove that Euler's function is multiplicative, i. e. for coprime numbers n *and* m

$$\varphi(nm) = \varphi(n)\varphi(m).$$

b)Prove the following identity:

$$\sum_{d|n} \varphi(d) = n.$$

c)Find a new proof of the fact that for $n = p_1^{\alpha_1} p_1^{\alpha_1} \dots p_k^{\alpha_k}$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right).$$

Problem 2. *Prove that a Möbius transformation in* $PGL_2(\mathbb{C})$ *can be obtained by first performing stereographic projection from the plane to the unit sphere, rotating and moving the sphere to a new location and orientation in space, and then performing stereographic projection (from the new position of the sphere) to the plane.*

Problem 3. *Suppose that a regular* $n-gon$ *with vertices* $A_1, \dots, A_n$ *is inscribed in a unit circle. Prove that*

$$\prod_{i=2}^{n} |A_1 A_i| = n.$$