Problem set 7

Exercise 1.
Let z and w be any two complex numbers. Prove that
1) $\Re(z\bar{w}) = \Re(w\bar{z})$,
2) $|z+w|^2 + |z-w|^2 = 4\Re(z\bar{w})$,
3) $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$,
4) Every complex number is a root of a polynomial with real coefficients of degree 2,
5) $|z| + |w| \geqslant |z+w|$.
6) Compute $(1+i)^{100}$.
7) Compute all powers of i.

Definition. *The affine ratio of three different numbers (points)* z_1, z_2, z_3 *is given by the following formula:*

$$[z_1, z_2, z_3] = \frac{z_1 - z_2}{z_3 - z_2}.$$

The cross-ratio of four different numbers (points) z_1, z_2, z_3, z_4 *is given by the following formula:*

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_2}{z_3 - z_2} : \frac{z_1 - z_4}{z_3 - z_4}.$$

Exercise 2. *a) Points* z_1, z_2, z_3 *lie on the same line if and only if there affine ratio is real.*
b) Points z_1, z_2, z_3, z_4 *lie on the same line or on the same circle if and only if there cross-ratio is real.*
c) Let z *be the point of the intersection of two tangent lines at* ζ *and* ζ' *to the circle of radius one with the center at* 0*. Prove that*

$$z = \frac{2}{\frac{1}{\zeta} + \frac{1}{\zeta'}}.$$

This expression is called the harmonic mean.

Problem 1 (Newton's theorem)**.** *The midpoints of the diagonals in a quadrilateral circumscribed about a circle are collinear with the center of the circle.*

Problem 2 (Gauss' Line)**.** *Consider an arbitrary quadrilateral* $ABCD$*. Let us denote the intersection point of the lines* AB *and* CD *by* E*, the intersection point of the lines* BC *and* AD *by* F*, the midpoint of* $[AC]$ *by* M*, the midpoint of* $[BD]$ *by* N*, and the midpoint of* $[EF]$ *by* O*. The the points* M*,* N *and* O *are collinear.*

Problem 3. *a) Find a factorization of the expression*

$$a^3 + b^3 + c^3 - 3abc.$$

b) Find a formula for the roots of a cubic equation

$$x^3 + px + q = 0.$$

Problem 4 (Sam Lloyd's 15 puzzle)**.**
Suppose that numbers $1, 2, 3, \ldots, 15$ *and "blank" are arranged in a* 4×4 *grid. A legal move is to swap the empty cell ("blank") with an adjacent cell. Prove that it is not possible to go through legal moves from*

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	

to

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

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