Problem set 6

Exercise 1. *Prove that the number of even permutations in $\mathbb{S}_n$ equals to the number of odd permutations.*

Definition. *Two elements g_1 and g_2 in the group G are conjugate if there exists an element $h \in G$, such that $g_1 = hg_2h^{-1}$.*

Exercise 2.
a) Prove that being conjugate is an equivalence relation.
b) Prove that for any group G and an element $g \in G$ the map

$$\varphi\colon G \longrightarrow G$$

given by the formula

$$x \longrightarrow gxg^{-1}$$

is an isomorphism. It is called an inner automorphism.
c) Prove that a composition of two inner automorphisms of G is an inner automorphism. Prove that inner automorphisms form a group.
d) Find the group of inner automorphisms of $\mathbb{S}_3$.

Exercise 3. *Prove that*
a) Similarities preserve angles and send circles to circles.
b) A composition of two homotheties with coefficients λ_1, λ_2 with $\lambda_1\lambda_2 \neq 1$ is a homothety with coefficient $\lambda_1\lambda_2$.
c) If a composition of three homotheties is the identity map, then there centers lie on the same line.
d) [**Monge's theorem**] *Outer tangent lines to the circles S_1 and S_2, S_2 and S_3, S_3 and S_1 intersect in the points A, B and C respectively. Prove that points A, B and C lie on the same line.*

Problem 1. *Prove that a group of orientation preserving symmetries of the dodecahedron is isomorphic to A_5.*

Problem 2. *Prove that two permutations π_1 and π_2 are conjugate in $\mathbb{S}_n$ f and only if for every k they have the same number of cycles of length k.*

Problem 3. *Let R_n denote a set of fixed point free permutations($\pi(i) \neq i$ for $1 \leqslant i \leqslant n$) in $\mathbb{S}_n$.*
a) Prove that

$$\sum_{\pi \in R_n} sgn(\pi) = (-1)^{n-1}(n-1).$$

b) Prove that

$$\lim_{n \longrightarrow \infty} \frac{|R_n|}{n!} = \frac{1}{e}.$$