Problem set 5

Exercise 1. *Consider a group G and its subgroup H. We call two elements x and y equivalent ($x \sim y$) if $xy^{-1} \in H$.*
1.Prove that $\sim$ is an equivalence relation. The corresponding classes of equivalence are called left cosets.
2.Prove that all left cosets have the same number of elements.
3.Prove that if $|G|$ is finite, then $|H|$ divides $|G|$.

Exercise 2. *Construct a surjective homomorphism from $\mathbb{S}_4$ to $\mathbb{S}_3$ (a surjective map $f\colon \mathbb{S}_4 \longrightarrow \mathbb{S}_3$, such that $f(\sigma\tau) = f(\sigma)f(\tau)$).*

Definition. *Suppose that $d|n$. Prove that the number of elements of order d in a cyclic group of order n equals to $\varphi(d)$. Here φ is the Euler function.*

Definition. *Consider a permutation $\sigma \in \mathbb{S}_n$. A number of inversions $Inv(\sigma)$ is the number of pairs $i < j$ such that $\pi(i) > \pi(j)$.*

Exercise 3.
1. Find the number of inversions in a cycle of length k.
2. What is the maximal number of inversions in a permutation in $\mathbb{S}_n$?

Problem 1. *Prove that*

$$sign(\sigma) = (-1)^{Inv(\pi)}.$$

Problem 2. *Prove that*

$$\sum_{\pi \in \mathbb{S}_n} x^{Inv(\pi)} = (1+x)(1+x+x^2)\dots(1+x+x^2+\dots+x^{n-1}).$$

Problem 3. *Prove that $\mathbb{S}_n$ is generated by $(12\dots n)$ and (12).*

Problem 4 (I). *Remove a corner from 101×101 chess board. Prove that the rest cannot be covered by triominoes. A "triomino" is like a domino, except it consists of three squares in a row.*

Problem 5 (Sylvester's theorem). *Consider n lines in the plane, not all of which pass through the same point. Prove that there exists a point in which exactly two of the lines intersect.*