Problem set 3

Exercise 1. *Prove that the sum of the elements on any diagonal of the Pascal triangle is a Fibonacci number:*

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} = F_{n+1}.$$

Definition. *An infinite product of numbers $a_1, a_2, a_3 \ldots$ is a formal expression $a_1 \cdot a_2 \cdot a_3 \cdot \ldots$ or $\prod_{k=1}^{\infty} a_k$. The value of it is defined as a limit of the sequence $P_n = a_1 a_2 \ldots a_n$. If this limit exists, the infinite product is called convergent.*

Exercise 2. *Suppose that $x_1, x_2, \ldots$ are positive. Prove that the infinite product*

$$(1 + x_1)(1 + x_2)(1 + x_3)\ldots$$

is convergent if and only if the series

$$x_1 + x_2 + x_3 + \ldots$$

is convergent.

Exercise 3. *Let p_n be the number of partitions of $n \in \mathbb{N}$ into positive integer summands (we don't distinguish between two partitions, which differ by the order of the summands only. For instance, $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, so $p_4 = 5$). Let q_n be the number of partitions of n, where all the summands are distinct (for instance, $q_4 = 2$). For $n = 0$, we put $p_0 = q_0 = 1$. Prove that*
a) the generating function $q(x) = \sum q_n x^n$ of the sequence $\{q_n\}$ is given by

$$q(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)\ldots;$$

b) the generating function $q(x) = \sum p_n x^n$ of the sequence $\{p_n\}$ is given by

$$p(x) = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)\ldots}.$$

Exercise 4. *Prove that a composition of three symmetries is a glide reflection.*

Exercise 5. *Suppose that G is a group and $S \subset G$ is a finite subset, such that if $x, y \in S$, then $xy \in S$. Prove that S is a subgroup of G.*

Problem 1. *The points $A_1, \ldots, A_n$ form a regular polygon, inscribed in a circle with the center O. A point X lies on the same circle. Prove that the images of the point X under the symmetries with axes $OA_1, OA_2, \ldots, OA_n$ form a regular polygon.*

Problem 2. *Consider n pairs of points on the segment AB, which are symmetric with respect to its center. Half of these points are red, the other are blue. Prove that the sum of distances from the point A to the blue points equals to the sum of distances from the point B to the red points.*

Problem 3. *Suppose that you have n pairs of parentheses and you would like to form a valid grouping of them, where "valid" means that every opening parenthesis has a matching closed one. For example $((())())$ is valid, while $())(())$ is not. What is the number of valid groupings?*