Problem set 1

Exercise 1. *Prove that*

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k},$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n},$$

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \ldots + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k},$$

$$\binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1} = \binom{n}{k},$$

$$\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + n^2\binom{n}{n} = ?.$$

Exercise 2 (**The Inclusion-Exclusion Principle**)**.** *Consider N objects and some list $P_1, P_2, \ldots, P_n$ of their properties. Let N_i be the number of objects satisfying P_i, N_{ij}, the number of objects satisfying P_i and P_j, and so on. Prove that the number of objects satisfying none of these properties is equal to*

$$N - \sum N_i + \sum_{i_1<i_2} N_{i_1 i_2} - \sum_{i_1<i_2<i_3} N_{i_1 i_2 i_3} + \ldots + (-1)^n N_{123\ldots n}.$$

Exercise 3.
1. Prove that the composition of a parallel transport and a central symmetry (in any order) is a central symmetry.
2. Prove that if one reflects a point symmetrically over points O_1, O_2, O_3 and then reflects it symmetrically over the same points once again, the point returns back to its initial position.
3. Consider three lines a, b, c on the plane. Let $F = S_a \circ S_b \circ S_c$. Prove that $F \circ F$ is a parallel transport.

Problem 1 (**Euler's function**)**.** *Let $n = p_1^{\alpha_1} p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the prime factorization of n and $\varphi(n)$ be the number of integers from 1 to n, which are coprime to n. Prove that*

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right).$$

Problem 2. *Consider two lines in the plane with the angle γ between them and suppose a grasshopper is jumping from one line to the other. Every jump is exactly 30 inches long, and the grasshopper jumps backwards whenever it has no other options. Prove that the sequence of its jumps is periodic if and only if $\frac{\gamma}{\pi}$ is a rational number.*

Problem 3. *Consider a regular polygon with vertices $A_1, A_2, \ldots, A_n$ and center O. Prove that*

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = 0.$$

Problem 4 (**I**)**.** *We will consider a "disease" that infects cells of a chessboard. Initially some number of the 64 cells are infected. Subsequently the infection spreads according to the following rule: if at least two neighbors of a cell are infected then the cell gets infected. (Neighbors share and edge, so each cell has at most four neighbors.) No cell is ever cured. What is the minimum number of cells one can initially infect so that the whole board is eventually infected? It is easy to see that 8 is sufficient in many ways. Prove that 7 is not enough.*